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VARIETIES OF MATHEMATICAL DISCOURSE IN PRE-MODERN SOCIO-CULTURAL CONTEXTS: MESOPOTAMIA, GREECE, AND THE LATIN MIDDLE AGES

AN HOMAGE TO DIRK STRUIK ON THE BEGINNING OF HIS TENTH DECADE

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AS HELLENES, YOU WILL KNOW THE PLATONIC VIEW that mathematical truths *exist*, and that they are eternal, unchanging, and divine.

As mathematicians you will, however, also know that mathematics is not adequately described as a collection of unconnected "truths"; the whole point in the activity of the mathematician, of the mathematics teacher, and of the applied mathematician for that matter, is the possibility to establish *connections* inside the realm of mathematics — between one theorem and another; between problems and theorems; between problems and procedures; between theorems, procedures and sets of axioms; between one set of axioms and another set, etc. — connections that in some sense (which I am not going to discuss here) map real

* The following is an elaborated and revised version of a paper presented to the seminar "Mathematics as a Science and as a School Subject" arranged by the Greek Mathematical Society, Branch of Thessaloniki, in April 1983. I have preferred to stick to the form of the original oral exposition, letting the text present the main argument and illustrations, and hiding away documentation, conceptual clarifications and qualificatory remarks in the notes.

It is a pleasure to express my heartfelt gratitude to the Greek Mathematical Society/Thessaloniki for inviting me as a lecturer to the seminar, and especially to its indefatigable and enthusiastic secretary, Nikos Kastanis, who arranged everything.

connections of the material or the human world.[1]

You will also know that such connections are established by means of proofs, demonstrations, *arguments*. Mathematics is a *reasoned discourse*.[2] Further, you will probably agree that the eagerness of the Ancients, not least Plato and Aristotle, to distinguish scientific argument from arguments concerned with mere opinion — the arguments of the rhetor and the sophist — undermines its own purpose: What creates the need for such eager distinctions if not the close similarity between the two sorts of arguments? On the other hand, you will also concede that *mathematical* discourse is often organized in agreement with the Aristotelian description of scientific argumentation, as argument from indisputable premises.

As teachers, finally, you will know that an argument — be it a mathematical argument — is no transcendental entity existing from before the beginnings of time. It is a human creation, building on presuppositions which in the particular historical (or pedagogical) context are taken for granted, but which on the other hand cannot be taken over unexamined from one historical (or pedagogical) situation to another. What was a good argument in the scientific environment of Euclid was no longer so to Hilbert; and what was nothing but heuristics to Archimedes became good and sufficient reasoning in the mathematics of infinitesimals of the seventeenth and eighteenth centuries — only to be relegated again to the status of heuristics in the mid-nineteenth century.[3]

1 See, e.g., Chandler Davis, "Materialist Mathematics," pp. 37–66, in R.S. Cohen, *et. al.* (eds.), *For Dirk Struik* (Dordrecht & Boston, 1974); B. Booss & M. Niss (eds.) *Mathematics and the Real World.* Proceedings of an International Workshop, Roskilde University Center (Denmark), 1978 (Interdisciplinary Systems Research, 68; Basel, Boston & Stuttgart, 1979); *Tekster fra IMFUFA, Roskilde Universitetscenter*, No. 18 (1979); and Dirk Struik, "Concerning Mathematics," *Science & Society*, Vol. I (1936–37), pp. 81–101.

2 Since the term "discourse" is used in slightly different ways by different authors, I shall betray the explanation which my translator in Thessaloniki forced out of me: "You will recognize that the modes of communication in school, in the Church, and in the Army are completely different; in the selection of themes or subjects about which one communicates, but in many other respects too. The didactical, ecclesiastical/religious and military *discourses* are different. Philosophically speaking, the discourse can be considered the *form* (in the Aristotelian sense) of the totality of communications of a certain institution or type of situation."

3 → Judith V. Grabiner, "Is Mathematical Truth Time-Dependent?", *American Mathe-*

So, one aspect of mathematics as an activity — other aspects I shall take for granted — is to be a reasoned discourse. The corresponding aspect of mathematics as an organized body of knowledge is to be the *product of communication by argument*, i.e., communication where a "sender" convinces by means of arguments a "receiver" that some statement or set of statements is true; in many cases the "receiver" is of course only a *hypothetical* average professional interlocutor of the mathematician, defined through the sort of arguments that are deemed adequate. Normally, either the truths communicated, or at least some broad base for the communication, is supposed to be fixed in advance; the discourse is not fully open. In principle this sort of communication can be described as *reasoned teaching*, the concept taken as a general philosophical category. So, teaching is not only the *vehicle* by which mathematical knowledge and skill is transmitted from one generation to the next; it belongs to the *essential characteristics* of mathematics to be *constituted through teaching* in this broad philosophical sense.

However, teaching, and even teaching regarded philosophically, belongs no more in the eternal Platonic heavens than do mathematical truths and arguments. Quite the contrary. Teaching is an eminently social activity, depending on context, personal and group characteristics of the persons involved, social norms, purpose of the teaching, material, cultural and linguistic conditions and means, etc. And so, if mathematics is constituted through being taught, we may expect it to be very much molded by the *particular* teaching through which it comes about.

It will be my aim in the following to trace that molding in the restricted case of *institutionalized teaching*. That is, I shall eschew the airy philosophical definition of reasoned teaching where the "sender" convinces the "receiver" by means of arguments organized inside a closed or semi-closed discourse; instead I shall concentrate on teaching as something involving a teacher and his students, ordered according to some fixed social and societal pattern, and normally taking place in a more or less formalized school. I shall stick to the pre-Renaissance (and so, pre-Gutenberg) era, and through three extended key episodes I shall

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matical Monthly, Vol. 81 (1974), pp. 354–365.

try to trace the relations between the development of mathematics, the character of mathematical discourse, and the institutional setting of mathematical teaching (mainly the teaching of adults).

I. *Mesopotamia: Scribal Computation, and Scribal School Mathematics*

The first of my extended episodes is the development of Sumerian and Babylonian mathematics, in a process of several phases covering some 1500 years. Parallel with the rise of the early proto-Sumerian city-states in the late fourth millenium B.C., a first unification of a variety of proto-mathematical techniques (for primitive accounting, practical geometry and measurement) into a single coherent system ("mathematics," in our terminology) united by the common application of numeration and arithmetic, appears to have taken place. The social environment of this unification was that of the Temple corporation — indeed, the institution which molded society into a *state* was, according to all existing evidence, the temple.[4] It seems, however, that mathematics did not grow out of the mere environment of administrators of temple property and taxation: various types of evidence suggest that the unification and coherence did not really correspond to practical administrative needs, which could have been provided for by isolated extensions of the existing separate techniques. Instead, like the development of more genuine writing, mathematical unification and coherence seem to be products of the school where future officials were trained, *and* where the techniques which they were going to apply were also developed.

The sources for this are few and scattered, and what I have just told is a reconstruction arising from the combination of many isolated pieces of evidence.[5] As time goes on, however, the picture becomes clearer. Toward the end of the third millenium

4 The ascription of a "molding" role for the proto-Sumerian temple does not imply acceptance of the traditional (Wittfogel-related) "Tempel-Stadt" thesis, according to which the Temple had complete societal dominance, and retained such dominance until c. 2400 B.C. → B. Foster, "A New Look at the Sumerian Temple State," *Journal of the Economic and Social History of the Orient*, Vol. 24 (1981), pp. 225–281; and Hans J. Nissen, "Die 'Tempelstadt': Regierungsform der frühdynastischen Zeit in Babylonien?", pp. 195–200 in H. Klengel (ed.), *Gesellschaft und Kultur im alten Vorderasien* (Berlin-DDR, 1982).

5 The detailed information on which I build this and most of my interpretation of Sumero-Babylonian mathematics is presented in my "Influences of Institutionalized Mathematics Teaching on the Development and Organization of Mathematical

B.C. Southern Iraq had been united into a single centralized royal state (the "Neo-Sumerian Empire" or "Ur III"), where the "Palace" directed large parts of the total economy through a vast bureaucracy.[6] This bureaucracy was carried by a body of *scribes*, who, at least since the mid-third millenium, had emerged as members of a specialized profession, and who had long since been taught in specialized schools. By the end of the third millenium, when sources describing the curriculum of the scribe school turn up, it is clearly dominated by applied mathematics.[7] At the same time, the professional ideology of the scribal craft (as inculcated in the scribal school) becomes visible in the sources: The scribe is, or is at least is expected to be, proud of his service to the royal state, which is presumed to serve general affluence and justice.

The centralized Neo-Sumerian state had a short life. After only a hundred years of existence it crumbled not least under the weight of its own bureaucratic structure. Still, it created two very important innovations in mathematics, of which one has served since then.

The first innovation is the introduction of *systematic accounting*, on occasion with built-in controls.[8] Such accounting systems

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Thought in the Pre-Modern Period," *Materialien und Studien. Institut für Didaktik der Mathematik der Universität Bielefeld*, Vol. 20 (1980), 7–137 (the Sumero-Babylonian development is discussed in pp. 14–29 and 80–94). References contained in that paper are repeated only to a restricted extent below.

6 The Mycenaean period of Ancient Greece offers an analogy, although on a more primitive and much smaller scale, as illustrated by the fact that writing was so exclusively reserved for Palace accounting that it was completely forgotten by the fall of the Palace economy. *Cf.* M.I. Finley, *Early Greece: The Bronze and Archaic Ages* (London, 1970), p. 63.

7 See p. 173 in A.W. Sjöberg, "The Old Babylonian Eduba," pp. 159–179 in *Sumerological Studies in Honor of Thorkild Jacobsen on his Seventieth Birthday, June 7, 1974* (The Oriental Institute of the University of Chicago, Assyriological Studies, No. 20; Chicago and London, 1976).

8 The current Neo-Sumerian "balanced account" did not possess the automatic built-in controls of later double-entry bookkeeping; but automatic control was known and used in certain cases; *cf.* pp. 147f. in V.V. Struve, "Some New Data on the Organization of Labour and on the Social Structure in Sumer during the Reign of the IIId Dynasty of Ur," pp. 127–171, in I.M. Diakonoff (ed.), *Ancient Mesopotamia, Socio-Economic History: A Collection of Studies by Soviet Scholars* (Moscow, 1969; a reprint of this highly interesting collection was published in Wiesbaden, 1973). In surveying too, built-in control was in use, probably as a check and in any case as a means to restrict the effect of measuring errors; *cf.* a field plan discussed in F. Thureau-Dangin, "Un cadastre chaldéen," *Revue d'Assyriologie*, Vol. 4 (1897), pp. 13–227.

were created or at least very suddenly spread in the administration of the whole empire[9] in a way which has rarely been equaled in history (the spread of double-entry bookkeeping, an analogous process in the Renaissance, took around 300 years). The second innovation (the one which was to survive) was the introduction of the *place-value system* for both integers and fractions — not with base 10 as in our "Hindu" system but with base 60 (as we still use it in the subdivisions of the hour and the angular degree taken over via Hellenistic astronomy). Even this system appears to have spread quickly, not for use in official documents but for the intermediate calculations of the scribes.

Both innovations built on foundations which had been laid during centuries of scribal school activity. Regarded from that point of view, they may be said to represent mere continuity. Still, mere continuity does not guarantee theoretical progress, and so both the occasions on which progress occurs and the precise character of the progress occurring should be noticed — in this case, the ability of the scribal school to respond to the demand created by a royal administrative reform, introducing well-functioning and sophisticated tools of applied mathematics. Another fact to be noticed is the seemingly total absence of mathematical extrapolation beyond the range of applications, in striking contrast to the tendencies of the following period. The scribal school and the scribal profession appear to have been so identified with their service function to the state that no interest in mathematics *as such* grew out of a curriculum of applied mathematics, in spite of clearly demonstrated mathematical abilities. The objective requirements and regularities of mathematical structures had created a tendency toward coherence (perhaps because of the way they manifested themselves when mathematics had to be made comprehensible in teaching?); but mathematical discourse and practice seem to have remained a non-autonomous, integrated part of administrative discourse and practice until the very end of the third millenium.

9 According to Hans Nissen of the Freie Universität Berlin (W), only scattered administrative texts have been found from the reign of the first king and from the earlier years of the second; then, all at once, thousands and thousands turn up, in a way which suggests a thorough administrative reform, organized via the scribal school (private communication).

This was to change during the following, so-called "Old Babylonian," period, c. 1900 B.C. to 1600 B.C. As already mentioned, the centralized Neo-Sumerian state crumbled under the weight of its top-heavy bureaucracy, and it was brushed aside by barbarian invasions. When organized states and civilized life had stabilized themselves once more, a new economic, social, and ideological structure had appeared. Large-scale latifundia had been replaced by small-plot agriculture, mostly held by tenants; the royal traders had become independent merchants; and the royal workshops had been replaced by private handicraft.[10]

This socioeconomic change was mirrored by social habits and ideology. In the Neo-Sumerian Empire, only official letter writing had existed; now, private and even personal letters appeared. The seal, once mainly a prerogative of the royal official, turned up as the private mark of the citizen. The Gods, with whom one had once communicated through the temples of the royal state, now took on a supplementary role as private tutelary Gods. And of course, the street scribe required to write personal letters on dictation appeared together with the free-lance priest performing private religious rites. All in all, the human being was no longer bound to the role of subject in the state (which he took on when losing his roots in the primitive community); he was *also a private man.* Most strikingly perhaps this is seen in the case of the King. He is still the state in person, but *at the same time* he is a private person, with his private tutelary God, who may differ from the tutelary God of the state.[11]

Through the scribal profession and the scribal school, all

10 *Cf.*, for instance, I.M. Diakonoff, "On the Structure of Old Babylonian Society," pp. 15–31 in H. Klengel (ed.), *Beiträge zur sozialen Struktur des alten Vorderasien* (Berlin-DDR, 1971); and pp. 52–83 in H. Klengel, *Hammurapi von Babylon und seine Zeit* (4th rev. ed., Berlin-DDR, 1980). Further details in my "Influences of Institutionalized Mathematics teaching . . . ," p. 87, n. 51.

11 For various aspects of this rise of the individual as a private person, *cf.* H. Klengel, "Zur Rolle der Persönlichkeit in der altbabylonischen Gesellschaft," pp. 109–117, in *Humanismus und Menschenbild im Orient und in der Antike.* Konferenzvorträge (Martin-Luther-Universität Halle-Wittenberg, Wissenschaftliche Beiträge 1977/28 (I, 2); Halle (Saale), 1977); *Hammurapi von Babylon . . . , op. cit.*, pp. 83–88; and F.R. Kraus, *Vom mesopotamischen Menschen der altbabylonischen Zeit und seiner Welt* (Mededelingen der Koninklijke Nederlandse Akademie der Wetenschappen, Afd. Letterkunde, nieuve Reeks Deel 36, No. 6; Amsterdam, 1973), *passim.* On religious change, see Th. Jacobsen, *The Treasures of Darkness: A History of Mesopotamian Religion* (New Haven and London, 1976), pp. 147ff.

this is reflected in the development of mathematics — such is at least my interpretaton of the great changes in mathematical discourse and practice. The scribes remained scribes, i.e., they continued to fill their old managerial and engineering roles; the school went on to teach them the accounting, surveying and engineering techniques needed for that; those activities and the school still imbued them with pride of serving as scribes for the royal state and the King, and so, supposedly, for general affluence and justice.[12] But the professional ideology of the scribes as revealed in school texts is not satisfied by mere usefulness. Even the scribal role has become a *private* identity. The scribe is proud of his scribal *ability* rather than of the functions which his ability permits him to carry out; and he is proud of a *virtuosity* going far beyond such abilities which would be functionally useful. A *real* scribe is not one who is able to read and write the current Akkadian tongue; these are presuppositions not worth mentioning. No, a scribe's cunning is demonstrated only when he is able to read, write and even speak the dead Sumerian language; when he is familiar with the argot of various crafts; and when besides the normal meanings of the cuneiform signs (a task in itself, since many signs carry both one or several phonetic meanings and one or several ideographic interpretations) he knows their occult significations. All these qualities, this virtuosity, had its own name: The scribes spoke of their special *humanity*. So, a human being *par excellence*, i.e., a *real scribe*, is one who goes beyond vulgar service, one whose virtuosity defies normal understanding. He is, however, still one whose virtuosity lies within the field defined by the scribal functions; the scribe can only be proud of *scribal* virtuosity.[13]

12 One text among many which identifies the King as the upholder of affluence and justice is Hammurabi's famous "law-code," more specifically its prologue and epilogue (translation, e.g., in J.B. Pritchard (ed.), *Ancient Near Eastern Texts Relating to the Old Testament* [Princeton, 1950], pp. 163–180). Recently, Marvin Powell has argued that the very rise of the royal State in the mid-third millenium should be sought in its mediating role in face of violent class conflicts inside the city-states (in a way reminiscent of Engels' analysis in *Der Ursprung der Familie, des Privateigentums und des Staates);* see p. 173 in his "Götter, Könige und 'Kapitalisten' im Mesopotamien des 3. Jahrtausends v.u.Z.," *Oikumene*, Vol. 2 (Budapest 1978), pp. 127–144. If Powell's assumption holds good, the "royal ideology" known to the scribes can be considered an embellishment of facts, but still a reflection of the real *raison d'être* of the state.

13 For this description of the Old Babylonian scribal culture, *cf.* Landsberger and others, pp. 94–123 in C. Kraeling and R. McC. Adams (eds.), *City Invincible, A Symposium on*

You may already have thought that mathematics must be an excellent tool for anybody needing to display special abilities beyond common understanding. And indeed, the so-called "examination texts," which permit us to decipher the scribal ideology, align mathematical techniques with occult writing and craftmen's argot.[14]

There is, however, more than this to the connection between the rise of private men, scribal "humanity," and mathematics. Indeed, an interest in mathematics beyond the purely useful develops in the Old Babylonian period.[15] It leads to large-scale creation of second- and higher-degree algebra. This algebra becomes a dominating feature of Old Babylonian mathematics; second-degree problems are found by the hundreds in the cuneiform texts of the period. Many of them look like real-world problems at first; but as soon as you analyze the structure of known versus unknown quantities, the complete artificiality of the problems is revealed — e.g., in the case where an unfinished siege-ramp is presented, of which you are supposed to know the total amount of earth required for its construction together with the length and height of the portion already built, but not the

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Urbanization and Cultural Development in the Ancient Near East (Chicago, 1960); F.R. Kraus, *Vom mesopotamischen Menschen der altbabylonischen Zeit* . . . , pp. 17–[illegible]; S.N. Kramer, "Schooldays: A Sumerian Composition Relating to the Education of a Scribe," *Journal of American Orential Studies*, Vol. 69 (1949), pp. 199–215; A. Falkenstein, "Die babylonische Schule," *Saeculum*, Vol. 4 (1953), pp. 125–137; A.W. Sjöberg, "In Praise of the Scribal Art," *Journal of Cuneiform Studies*, Vol. 24 (1971–72), pp. 126–129; "Der Vater und sein missratener Sohn," *Journal of Cuneiform Studies*, Vol. 25 (1973), 105–169; "Der Examenstext A," *Zeitschrift für Assyriologie und Vorderasiatische Archäologie*, Vol. 64 (1975), pp. 137–176; and "The Old Babylonian Eduba" (above, n. 7).

14 See Landsberger, pp. 100f. in Kraeling and McC. Adams, *City Invincible*, and Sjöberg, "Der Examenstext A."

15 The Mesopotamian written record is full of random lacunae; still the beginnings of the new style of Mesopotamian mathematics in the Early Old Babylonian period appears to be established beyond reasonable doubt. For this, statistical arguments can be given (see my "Influences . . . " p. 87). Further, the formulation in the "new" Akkadian literary language is important evidence: the old literary traditions were transmitted for almost two millenia in Sumerian, *cf.* W.W. Hallo, "Toward a History of Sumerian Literature," in *Sumerological Studies in Honor of Thorkild Jacobsen* (above, n. 7), pp. 181–203. Finally, the basic technique (*viz.* the quadratic completion) for the treatment of second-degree equations which form the backbone of the "new style," seems to have carried the name "the Akkadian Method," according to hitherto unnoticed evidence in a Late Old Babylonian tablet, *cf.* pp. 95f. in my discussion paper, "Babylonian Algebra from the View-Point of Geometrical Heuristics" (Roskilde University Center, 1984).

total length and height to be attained.[16] These algebraic problems can be understood, and can in my opinion only be understood, as being on a par with correct Sumerian pronunciation and familiarity with occult significations of cuneiform signs: scribal ability, true enough, but transposed into a region of abstract ability with no direct practical purpose. As scribal discourse in general, mathematical discourse has been disconnected from immediate practice; it has achieved a certain autonomy.

Thus, Old Babylonian "pure mathematics" must be understood as the product of a teaching institution which no longer restricted itself to teaching privileged subjects what they needed to perform their future function as officials — i.e., it can be understood as the product of an institution where teachers and students (especially perhaps the teachers[17]) were also persons with a *private* identity as scribes.

Another aspect of Old Babylonian "pure mathematics" can, however, only be understood if we see it as a product of a *scribal* school. This aspect is the fundamental difference between Old Babylonian and Greek "pure mathematics." I used the term "virtuosity" to describe the scribal "humanity"; it is also an adequate description of Old Babylonian mathematics. Old Babylonian mathematics grew out of its *methods*, while Greek mathematics grew out of *problems*, to state things very briefly.

16 The problem occurs in two different texts, BM 85194, Rev. II, 22–33, and BM 85210, Obv. II, 15–27 — incidentally solved by two different methods. A third still more artificial problem concerning the same ramp is BM 85194, Rev. II, 7–21. All three will be found in transliteration and translation together with a discussion in O. Neugebauer, *Mathematische Keilschrift-Texte* I (Berlin, 1935; reprint Berlin-W and New York, 1973). *Cf.* also my "Babylonian Algebra . . . ," pp. 53–61.

17 In the Middle and Late Old Babylonian "examination texts," "the whole character of the dialogue changes. The master becomes more boastful, and the examinee assumes a merely secondary role of giving the master an opportunity to display his knowledge" (Landsberger, in Kraeling and McC. Adams, *City Invincible*, p. 112). In the mathematical domain, especially the higher-degree equations can be taken as examples of the same phenomenon: No general methods for the solution of such equations were known. As pointed out by Thurea-Dangin, the Babylonians "demonstrate their inability to solve the third-degree equation" by the very methods they employ to do so (*Textes mathématiques babyloniens* [Leiden, 1938], p. xxxviii). So, the higher-degree equations which occur are all constructed so as to be solvable by a special trick — a trick which no student would be able to transfer to analogous problems lacking the specially prepared coefficients. Such problems do not train the abilities of the student; they display the fake abilities of the teacher.

This may sound strange. Indeed, Babylonian mathematics is known only from problems; it does not contain a single theorem and hardly a description of a method.[18] That, however, follows from the training role of the Babylonian school, which did not aim at the theoretical understanding of methods but at the *training of methods* — first, of course, the training of methods to be used in practice, but next also of methods which would permit the solution of useless second-degree problems. Such training could only be obtained through drill. Indeed, the problems which occur appear to be meant exactly for drill; at least, many of them appear to have been chosen not because of any inherent interest but just because they *could be solved* by the methods at hand.

Greek mathematics, on the other hand, for all the theorems it contains, grew out of problems, for the solution of which methods often had to be created anew. We may think of the Squaring of the Circle, the Trisection of the Angle, and the Delic Problem (Doubling the Cube). But we should not forget the Eudoxean theory of proportions or book X of the *Elements* on the classification of irrationals and the resulting algebraic relations between the resulting classes.[19] They too are investigations of problems — problems arising from and only given meaning through the development of mathematical theory.

The Greek effort to solve fundamental problems is clearly related to the whole effort of Greek philosophy to *create theoretical understanding*. That could never be the aim of a scribal school. There, skillful handling of methods was the central end. Problems were the necessary *means* to that end, and in the Old Baby-

18 In fact, two Old Babylonian texts appear to exhibit attempts to describe a method in general terms: Im 52301, Edge, and AO 6770, Obv. 1–8; see, e.g., K.-B. Gundlach and W. von Soden, "Einige altbabylonische Texte zur Lösung 'quadratischer' Gleichunger," *Abhandlungen aus dem mathematischen Seminar der Universität Hamburg*, Vol. 26 (1963), pp. 248–263; S. Brentjes and M. Müller, "Eine neue Interpretation der ersten Aufgabe des altbabylonischen mathematischen Textes AO 6770," *NTM. Schriftenreihe für Geschichte der Naturwissenschaften, Technik und Medizin*, Bd. 19 Heft 2 (1982), pp. 21–26. Both demonstrate by their clumsy and opaque formulations that abstract explanation of methods was something utterly unfamilar and difficult for their authors.

19 The term "algebraic" for once taken in the sense given to it in 20th-century mathematical theory.

lonian school they became the necessary *pretext* for the display of virtuosity.[20] Paganini, not Mozart, plays the violin.

Mutatis mutandis, we can speak of a Babylonian parallel to the effects of the "publish-or-perish" pressure on contemporary publication patterns. When submitted to such pressure, the easiest way out is to choose your problems according to their accessibility, given your ability and the methods with which you are familiar; current complaints that the scientific literature is drowned in publications treating problems of no other merit than that of treatability indicate, even if great exaggeration is allowed for, that some researchers *have* found this way out.[21]

Apart from its determination by methods rather than problems, another characteristic of Old Babylonian "pure mathematics" seems to derive from its particular background: *viz.*, that you have to analyze the structure of its problems in order to decide whether they are practical or artificial, i.e., "pure" (*cf.* the siege-ramp-problem mentioned above; superficially regarded, it looks like a typical piece of militarily applied mathematics). In other words, even when Old Babylonian mathematics is "pure" *in substance*, it remains *applied in form*. In contradistinction to this, the prototype of Greek mathematics is pure in form as well as in substance, to such an extent that the applications of geometry to astronomy are formulated abstractly as dealing with spheres in general — so in Autolykos' *Moving Sphere* and Theodosios' *Spherics*.[22] Greek mathematics had become pure in form, i.e., fully abstract, even when it was applied in substance.

This "applied form" of Old Babylonian mathematics could

20 Strictly speaking, two types of problems should be distinguished, *cf.* above, n.17: The huge lot of problems which train the solution of first- and second-degree equations and problems; and the much fewer problems of a higher degree, which are constructed so as to be solvable by special tricks. The former train methods, which may be of no practical value, but which may then serve the display of virtuoso ability; the latter train nothing at all — they *are* a means of display. In both cases, the methods (and tricks) at hand determine which problems shall and can occur.

21 At least in Denmark, such complaints are mainly directed against medical research. This drives the parallel even further, since it is precisely the medical establishment which is characterized by emphasizing publications when choosing candidates for promotion, without being really able to evaluate the scientific content or importance of specialized research. Publications therefore serve precisely as a means to display professional ability.

22 Autolykos, *Rotierende Kugel* and *Aufgang und Untergang der Gestirne*. Theodosios, *Sphaerik: Übersetzt und mit Anmerkungen versehen von Dr. Arthur Czwalina* (Ostwalds

hardly have been different; at least, it is in full harmony with other aspects of the scribal "humanity." The virtuosity which a scribe could covet had to look like *scribal virtuosity*. The scribe, however, was an *applier of mathematics*, a *calculator*; there were no social sources, and no earlier traditions, from which a concept of mathematics as an activity *per se* could spring, and there was thus no possibility that a scribe could come to think of himself as *a virtuoso mathematician*. Only the option to become a *virtuoso calculator* was open; so, Babylonian "pure mathematics" was in fact calculation pursued as *art pour l'art*, mathematics applied in its form but disengaged from real application. Expressed in other terms, Old Babylonian mathematical discourse had achieved autonomy for its actual working; it remained, however, *defined* through scribal professional practice.

So much about Sumerian and Babylonian mathematics. The argument could be supported by telling the story of Babylonian mathematics after the disappearance of the individualized Old Babylonian social structure and of the scribal school as an institution — "pure" mathematics vanishes from the sources for more than a thousand years — or by comparing Egyptian and Babylonian developments. This, however, I shall bypass,[23] and I shall close my first episode by emphasizing that the overall social characteristics of the institution in which Mesopotamian mathe-

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Klassiker der Eksakten Wissenschaften 232; Leipzig, 1931). Interestingly, the Islamic astronomers were dissatisfied with this formal purity of Greek spherics. In most Islamic paraphrases of the Greek works, commentaries are inserted explaining what the theorems are *really* about in astronomy; this is discussed in G.P. Matvievskaya, "On Some Problems of the History of Mathematics and Astronomy of the Medieval East," pp. 21–31 in *XVIth International Congress of the History of Science, Bucharest 1981. Papers by Soviet Scientists: The 1000th Anniversary of Ibn Sina's Birth (Avicenna)* (Moscow, 1981). In the Latin Middle Ages, the same thing came to happen: The formally pure 13th century mathematician Jordanus of Nemore wrote a treatise on the stereographic projection in the Greek style; it achieved a rather wide circulation, but only in versions with extra commentaries explaining the astronomical substance of the work; see *Jordanus de Nemore and the Mathematics of Astrolabes: De Plana Spera.*.An edition with introduction, translation and commentary by Ron B. Thomson (Toronto 1978), and pp. 27f. in my "Jordanus de Nemore: A Thirteenth Century Mathematician in and out of Touch with the Intellectual Currents of His Times" (roneotyped contribution to the XVIth Int. Congress Hist. Science; Roskilde University Center, 1981). Evidently, the formal purity of substantially applied mathematics must be considered a special characteristic of Greek mathematics: in other cultures it was intentionally avoided.

23 A brief exposition of both subjects will be found on pp. 27–36, 92–97, in my "Influences . . . " (*cf.* above, n. 5).

matics was taught and developed influenced primarily the overall formal characteristics of Mesopotamian mathematics as a discourse and as a developing system, and normally *not* its factual contents. The Babylonian and the Greek would calculate the diagonal of a rectangle with sides 5 and 12 in the same numerical steps; so far, of course, mathematics *does* consist of socially and historically transcendental truths.

II. *Greek Mathematics: From Open Reasoned Discourse to Closed Axiomatics*

My next episode will be the development of Greek mathematics. While the role of argumentation in the Mesopotamian development follows mainly from indirect evidence supplemented by a few clay tablets where something like a didactical explanation occurs,[24] Greek mathematics is argument through and through.

Greek mathematics is the product of "reasoned teaching" in the general philosophical sense which I gave to that expression. It is more dubious whether it is a product of institutionalized teaching; it may be that the basic reason for the overwhelmingly argumentative character of Greek mathematics should be sought in its initial *lack* of school institutionalization.

At first, however, a very different hypothesis suggests itself. Much of Greek mathematics has been seen as a search for harmony and completeness — and do these ideals not look as if they were taken over from the ideals of the *paideía eleúthera*, the liberal education of the free citizen as a harmonious and complete human being?[25] Furthermore, once institutionalized (from the fourth century B.C. onwards) the *paideía* came to contain a fair measure of mathematics. Finally, Proklos tells us that Pythagoras

24 Texts IX and XVI in E.M. Bruins and M. Rutten, *Textes mathématiques de Suse* (Mémoires de la Mission Archéologique en Iran, XXXIV; Paris, 1961), according to a new interpretation of these texts; see my "Babylonian Algebra . . . ," pp. 86–98. In the same paper it is also shown, from an investigation of terminology, formulations and concepts, that the Old Babylonian algebraic texts rest on a basis of mostly geometric justification; they are not just recipe-descriptions of algorithms, as is often inferred from misleading translations.

25 Without being so formulated, this is the conception behind the prominent place of mathematics in the classically oriented neo-humanist educational ideal of 19th and earlier 20th century Germany.

gave mathematics the "character of a liberal education," *schêma paideías eleuthérou.*[26]

On closer investigation, however, there seems to be no causal chain leading from the *paideía* to mathematics. Apparently, we are confronted partly with two analogous but distinct structures, partly with a causal chain leading the opposite way, partly perhaps with a neo-humanist optical illusion. Therefore, instead of analogies, I shall try to build my exposition on the actual evidence,[27] incomplete as it is.

I shall try to distinguish three periods, of which those of most critical interest for the investigation are unhappily the most difficult to document:

• The rise of reasoned mathematics, in the sixth and fifth centuries B.C. — "pre-Socratic mathematics";

• The creation of deductive and axiomatic mathematics, in the fourth and early third century — from Plato and Eudoxos to Euclid.

• And finally the mature period from Euclid onwards, in which the style and character of Greek mathematics was already fixed, and in which every Greek mathematical text known to us, a few fragments aside, was created.[28]

The first period is that of the Pythagorean order, the philosophical schools, and the sophists. Of these, the sophists, who

26 G. Friedlein (ed.), *Procli Diadochi in Primum Euclidis Elementorum Librum Commentarii* (Leipzig, 1873), 65, p. 16f.; translated in G.R. Morrow (transl., commentary), Proclus, A *Commentary on the First Book of Euclid's Elements* (Princeton, 1970). Since Morrow indicates the Friedlein-pagination, further quotations from Proklos will refer only to the Friedlein edition.

Of course, Proklos should not be considered as serious a source as is sometimes done. In itself, a commentary written in the fifth century A.D. is late; furthermore, Proklos the neo-Platonist is biased in his sympathy for Pythagoreans and for neo-Pythagorean sources; finally, concerning this specific quotation, the expression "liberal education" is suspicious as a clear anachronism in relation to Pythagoras' time and ideals. In fact, I am told by Harald Boehme that Eva Sachs has shown the passage to be a forgery due to the neo-Pythagorean Iamblichos (private communication).

27 Once more, a more detailed discussion of this evidence will be found in my "Influences . . . " (above, n. 5), pp. 37–52, 97–108.

28 "Was created," at least in the sense that it acquired its final form and place; much in Euclid's *Elements* was taken over in only partially reworked form from earlier works and traditions; W. Knorr argues that book X may have been taken over in virtually unchanged form; see "'La croix des mathématiciens': The Euclidean Theory of Irrational Lines," *Bulletin of the American Mathematical Society*, N.S., Vol. 9 (1983), pp. 41–69.

Euclid should not be taken quite seriously as a watershed. His *Elements* play a deciwere creators of a theoretical concept of education, and whose *paideía* aimed exactly at the creation of the complete human being or citizen, have the least to do with mathematics. Truly, Hippias, Bryson and Antiphon interested themselves in the trisection and circle-squaring problems. But Hippias' curve for trisection[29] is nothing but a smart trick, and Bryson's and Antiphon's treatments of the squaring shows them (according to Aristotle's polemics) to lie outside the mainstream of Greek mathematical thought.[30] One is tempted to assume that the sophists' treatment of these problems reflects the necessity for those professional teachers to deal with mathematical subjects *à la mode* in order to satisfy their clientele. If such is the case, sophist education underwent the influence of mathematics, not the reverse.

In the case of the Pythagorean order,[31] it is difficult to distinguish legend from history. It seems sure, however, that the Pythagorean movement was the place where *mathēmata* changed its meaning from "doctrines," i.e., "matters being learned," to "knowledge of number and magnitude," i.e., to "mathematics,"[32]

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sive role; but he also happened to have lived (we assume) at a time when things were anyhow taking place; *cf.* G.Aujac, "Autolycos de Pitané, prédecesseur d'Euclide," *Cahiers du Séminaire d'Histoire des Mathématiques*, Vol. 5 (Toulouse, 1984), pp. 1–12, where wide-range codification down to single formulations is shown to have occurred by the late fourth century B.C.

29 The *quadratrix*, named so in later times when it was also used for the squaring (or, better, the "rectification") of the circle. See T.L. Health, *A History of Greek Mathematics* (I-II; Oxford, 1921), Vol. I, pp. 226–230; Bulmer-Thomas, "Hipias of Elis," Vol. VI, pp. 406–408, *Dictionary of Scientific Biography* (I-XVI; New York, 1970–1980; hereafter DSB).

30 Aristotle, *Analytica priora* $75^{b}40$–$76^{a}3$, and *De sophisticis elenchis* $171^{b}16$–22, $172^{a}3$–7 (here and everywhere I use *The Works of Aristotle*. Translated into English under the editorship of Sir David Ross. I–XII. Oxford, 1908–1952). *Cf. Greek Mathematical Works* (I–II, Loeb Classical Library 335, 362; London & Cambridge, MA, 1939, 1941), Vol. I, pp. 310–317; G.B. Kerferd, "Antiphon," in DSB I, pp. 171f.; and Ph. Merlan, "Bryson of Heraclea," in DSB II, pp. 549f.

31 As a background to his own personal conclusions, with which one may agree or disagree, B.L. van der Waerden offers a balanced presentation of the whole available source material in *Die Pythagoreer. Religiöse Bruderschaft und Schule der Wissenschaft* (Zürich and München, 1979). *Cf.* however, the cautious attitude of P.-M. Schuhl concerning the range of early Pythagorean mathematical knowledge in *Essai sur la formation de la pensée grecque* (rev. ed., Paris, 1949), pp. 257–260.

32 See Archytas, fragment B.1 in H. Diels, *Die Fragmente der Vorsokratiker*, herausgegeben von W. Kranz (I–III; reprint of 6th ed., Dublin & Zürich, 1972), Vol I., pp. 431–435.

no later than the late fifth century B.C. It is well established that an essential part of the teaching of the order, be it secret or not, was dedicated to theoretical arithmetic, to geometry, to harmonics, and to astronomy — the four *mathēmata* listed by Archytas. Finally, their active role in the development of arithmetic, harmonics and astronomical speculation during the fifth century can be trusted with confidence.[33]

According to tradition,[34] the order consisted of an inner and an outer circle, *mathēmatikoí* and *akousmatikoí*, "mathematicians" and "listeners" — if the two groups were not a result of a split in the order. The latter are supposed to have been literal followers of the tradition, while the former are supposed to have been taught rationally, and perhaps to have made rational inquiry (a supposition which is independent of the question whether their group went back to Pythagoras himself, or was a later fabrication). If this tradition is reliable, the teachings of the *akousmatikoí* will at most have contained semi-mystical numerology. Some Pythagoreans may, on the other hand, have started rational inquiry from this basis; and their investigations of theoretical arithmetic, of harmonics, and of the problems of the irrational, may have become part of a cumulative research tradition *because of* integration with a stable tradition for reasoned teaching of the *mathēmatikoí*. More than this can hardly be said, given the lack of adequate sources.[35]

Still, I find it doubtful whether the rational and abstract character of Greek mathematics can have *originated* inside the circle of *mathēmatikoí*. It would seem to be in better harmony with the picture of the Pythagorean order as a mystical, religious and ethico-political movement, if available philosophical knowl-

33 More subject to doubt is the traditional ascription of the geometrical "application of areas" and the whole "geometric algebra" to the Pythagoreans. See W. Knorr, *The Evolution of the Euclidean Elements* (Dordrecht and Boston, 1975), pp. 199f., and, in contrast, van der Waerden's review of that book in *Historia Mathematica*, Vol. 3 (1976), pp. 497–499.

34 The (partly incoherent) evidence in Ancient sources is exposed in van der Waerden, *Die Pythagoreer*, pp. 64ff.

35 Except, perhaps, that the cumulation seems not to have gone very far in the later fifth century, if we are to judge from the fragments which can be ascribed to the Pythagorean Philolaos; see Kurt von Fritz, "Philolaus of Crotona," DSB X, pp. 589–591. This agrees with the most plausible conjecture, that the *mathēmatikoí* are a secondary formation (*cf.* v.d. Waerden, *op. cit.*, pp. 68ff.), and that serious accumulation had only begun by the mid-fifth century.

edge, including mathematics, was borrowed from outside in cases where it could serve the overall world view and aim of the movement[36] — possibly in a process of several steps, where the most elementary abstract arithmetic was taken over (as numerology and for use in musical and cosmological speculation) in the late sixth or early fifth century, and more advanced subjects and methods during the run of the fifth century. In fact, various sorts of evidence point in that direction.[37] It thus appears that the mathematical activity of the Pythagoreans consisted of work in agreement with the rational tradition when it was already established, and refinements of the same tradition. Therefore, I

36 Cf. Schuhl, *Essai sur la formation de la pensée grecque*, pp. 242–257.

37 Firstly, a (probably authentic) fragment from a comedy written by the poet Epicharmos at latest in the earlier part of the fifth century shows that some sort of abstract discussion of odd and even numbers was supposed to be well known to the general public (Diels-Kranz, *Fragmente der Vorsokratiker* I, p. 196). This agrees only badly with the idea that the doctrine of odd and even was created from scratch by the *mathēmatikoi*, select members of a Pythagorean Inner Party (or for that matter by Pythagoras himself). The claim that Epicharmos himself should be a Pythagorean appears to be ill-founded; cf. K. Freeman, *The Pre-Socratic Philosophers. A Companion to Diels, Fragmente der Vorsokratiker* (3rd ed., Oxford 1953), pp. 132–135.

Secondly, the Epicharmos-fragment shows beyond doubt that the early doctrine of odd and even built on arrangements of *psêphoi*, stone calculi. *Cf.* also W. Lefèvre, "Rechensteine und Sprache," pp. 115–169 in P. Damerow and W. Lefèvre, *Rechenstein, Experiment, Sprache* (Stuttgart, 1981); Knorr, *The Evolution of the Euclidean Elements*, pp. 135ff.; O. Becker, "Die Lehre vom Geraden und Ungeraden im Neunten Buch der Euklidischen Elemente," *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik, Abteilung B*, Vol. 3 (1936), pp. 535–553. The particular contribution of the Pythagoreans to arithmetic appears, on the other hand, to start from the connections between arithmetic and harmonics, as revealed in the theory of ratio and proportion; cf. Á. Szabó, *Anfänge der griechischen Mathematik* (München, Wien and Budapest, 1969), pp. 136–156; K. von Fritz, *Grundprobleme der Geschichte der antiken Wissenschaft* (Berlin-W and New York, 1971), pp. 47–52. Such a start agrees well with the ethical and religious attitude of the Pythagoreans, but it presupposes the disposability of an arithmetic which permits and invites one to discover the arithmetical properties of musical harmony.

Thirdly, Szabó argues that the basic characteristics of Pythagorean arithmetic — its distinction between "one" and "numbers," and its replacement of fractions by the theory of proportions — must be understood as results of a reformulation of the theory of numbers on Parmenidean and Eleatic foundations (*op. cit.*, pp. 352–361). This, however, would place its development well after the testimony of Epicharmos, and agree with a gradual mathematization of the doctrine influenced by the developments of philosophy and mathematics outside the order. Even the indirect proof, a favorite technique in the supposedly Pythagorean arithmetic included in the *Elements* would be a borrowing from Eleatic dialectics, as argued by Szabó (*op. cit.*, pp. 328–342) and G.E.R. Lloyd, *Magic, Reason and Experience: Studies in the Origins and Development of Greek Science* (Cambridge, 1979), p. 110. So, it cannot have been used in earlier Pythagorean investigations of the odd and the even.

shall direct attention to the open philosophical "schools."

Here, a word of caution may be appropriate. The philosophers' "schools" were probably not schools in an institutional sense. Even when we go to the mathematical "schools" of the fourth century, what translators designate, e.g., "the school of Menaechmos" is spoken of by Proklos simply as "the mathematicians around Menaechmos."[38] No doubt the philosophers had disciples; but they are distinguished from the sophists precisely by not being *determined* as professional teachers.[39] The philosophers made rational inquiry, some of them in mathematics; and they taught. Both activities must be understood as implying rational discourse. But nothing indicates that the philosophers' inquiry was determined in style and structure by their teaching.

Instead, the reasoned and abstract structure of fifth-century mathematics and its orientation toward *principles* must be sought elsewhere. That such a tendency was there is obvious even from the scant source material at our disposal, be it Hippocrates' investigation of lunes[40]; his writing of the presumably first set of *Elements*[41]; Oenopides' presumed first theoretically founded *construction* of dropping and ascending perpendiculars and his singling out of ruler and compass in that connection[42]; the description of the solar movement in the ecliptic as an inclined great circle (equally ascribed to Oenopides[43]); or a number of Platonic passages, from the references to investigations of in-

38 Friedlein, *Procli Diadochi in Primum Euclidis* . . . , p. 789. Although Proklos is in general a problematic source, as pointed out above, n. 26, he must be considered relatively reliable for the fourth century B.C., where he builds on the first-hand knowledge of Aristotle's co-worker and disciple Eudemos.

39 Conversely, Plato's polemics against the sophists are built on the fact that they live not *for* philosophy, as philosophers who incidentally teach, but *off* philosophy as paid teachers; *cf. Protagoras* 313c–314b (here and everywhere, I use Platon, *Oeuvres complètes*. Traduction nouvelle et notes par Léon Robin. I–II. Paris, 1950).

40 See the fragment in *Greek Mathematical Works* I, pp. 234–252. *Cf.* Heath, *A History of Greek Mathematics* I, pp. 183–200.

41 Friedlein, *Procli Diadochi in Primum Euclidis* . . ., p. $66^{7\text{-}8}$.

42 See pp. 180f. in I. Bulmer-Thomas, "Oenopides of Chios," DSB X, pp. 179–182; and Szabó, *Anfänge* . . ., pp. 369–373, commenting upon Proklos referring to Eudemos (Friedlein, *Procli Diadochi* . . ., $283^{7\text{ff.}}$, $333^{5\text{f.}}$).

43 Cf. Bulmer-Thomas, "Oenopides of Chios," p. 179. Bulmer-Thomas claims that the Ancient traditions must refer to a determination of the obliquity of the ecliptic, since "the Babylonians no less than the Pythagoreans and Egyptians must have realized from early days that the apparent path of the sun was inclined to the celestial equator." However, neither Babylonian nor Egyptian astronomy would describe the eclip-

commensurability[44] to the slave guided by suggestive questions to the doubling of a square.[45]

It is also evident that all these pieces of evidence point toward various locations inside the "philosophical movement." The immediate background to the rise of reasoned mathematics is thus the condition of mathematical discourse and investigation as part of a general philosophical discourse.

This observation is also in harmony with chronology. Even though a number of mathematical discoveries are ascribed to Thales, and such ascriptions can neither be proved nor disproved (not least because it is not clear what precisely Thales is supposed to have discovered), the philosophical transformation of myth and cosmogony into philosophical cosmology seems to precede the rise of genuine reasoned mathematics. Indeed, even the Eleatic critique of natural philosophy appears to precede, if not the first steps toward a reasoned approach to arithmetic, then at least the techniques of proof which came to characterize Greek mathematics as we know it.[46]

So, the rationality of fifth-century Greek mathematics appears to build on general philosophical rationality, and on an open, non-hierarchical type of discourse: not the one-way master-student relationship of institutionalized teaching but a discourse of mutual disagreement, conflict and common search.[47] While the former may be more effective for the assimilative expansion of a knowledge system, the latter, open discourse, may be the presupposition for fundamental change.[48] So, if I am right in my interpretations, the *specific* formation of

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tical movement as a movement in a *plane* cutting the celestial *sphere* in an inclined *great circle*; so, the earliest reference to Oenopides' discovery (*Erastae* 132a–b, a pseudo-Platonic dialogue from c. 300 B.C.) suggests that it was precisely the coupling of astronomical phenomena to abstract geometrical description.

44 *Theaetetus*, 147d–148b; *Laws*, 910a.

45 *Menon*, 82b–85b.

46 *Cf.* above, n. 37.

47 The counterposition of the two was made already in Antiquity. The Socrates of Plato's early writings discusses both the sophist master-student relation, where ready-made knowledge is poured into the mind of the defenseless disciple (*Protagoras*, 314b) and the non-hierarchical relationship practiced by Socrates himself, who was never the teaching master of anybody (*Apology*, 33a–b).

48 The reader may observe a certain convergence with the Kuhnian distinction between "normal science" and "scientific revolution," and with the Piagetian distinction between the quiet periods of (infant) cognitive development dominated by assimilation,

Greek mathematics may have originated in the *lack* of didactical institutionalization of the soil from which it grew during the first phase, as claimed above.

The open-type discourse of the philosophical environment may have had wider, social backgrounds: perhaps not so much in fixed social institutions as in the break-up of institutions. Indeed, the Solon reforms, which averted social conflict by instituting reasoned constitutional change, are contemporary with the earliest Milesian philosophy — and they are not the first attempt of their kind.[49] About a century before Solon and Thales, Hesiod presents us with an instance of conceptual analysis by dichotomy,[50] in a way which reminds one very much of Plato's dialogues, but which in historical context shows that the germs of logical analysis are older than philosophy in Greece. Still earlier, at the dawn of Greek literature, the rhetoric of the Homerian heroes contains clear dialectical, syllogistic figures (used even to persuade the Gods).[51] Ultimately, the discourse of early Greek rational philosophy may go back to the open discourse of the Popular Assembly and the *agora*[52]; while, as we saw, the discourse of Mesopotamian mathematics, *explaining* procedures and *training* scribes rather than *investigating problems* or questioning, goes back to the more closed discourse of organized school teaching.[53]

The second phase of the development of Greek mathemat-

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and the transitions from one cognitive stage to another. The convergence is not accidental. Nor is it, however, complete.

49 *Cf.* G. Smith, "More Recent Theories on the Origin and Interrelation of the First Classifications of Greek Laws," *Cahiers d'Histoire Mondiale*, Vol. 3 (1956–57), 173–195.

50 In *Works and Days*, verses 11ff., a discussion of two different sorts of "struggle" (*éris*). See Hésiode, *Théogonie — Les travaux et les jours — Le bouclier*. Texte établi et traduit par Paul Mazon (5th ed., Paris, 1960).

51 *Odyssey* IV, 962–766 is one clear instance. In *Odyssey* I, 60–62, a related argument is used by Athena in the Assembly of the Gods (Loeb Classical Library; London and Cambridge, MA, 1966; 1st ed., 1919).

52 I shall leave aside the connections between the particular coloring of Greek philosophical rationality (*viz.* the quest for *pure* and the widespread neglect of "productive" knowledge); the prevailing contempt for manual work; and the spread of slavery.

53 This counterposition is not meant as a full support to the Ancient Greek counterposition of the "democratic spirit" of the Greek city-state and the "despotism" of the Orient; see, in addition to Herodotos' *Histories, passim*, and Aristotle's *Politica*, 1285^a16ff. and 1327^b26ff., the constrasting passages in Aeschylus' *The Persians*, 211–214; *The Suppliant Maidens*, 370–401; and *Seven Against Thebes*, 6-9 (Loeb Classical

ics, going from Archytas, Plato and Theaetetos, not only brings a marked quantitative growth of mathematical knowledge, ex-

Library; London and Cambridge, MA, 1973; lst ed., 1922). At least in the earlier third millenium, a "primitive democracy" with popular assemblies existed in the Sumerian city-states (*cf.* Th. Jacobsen, "Early Political Development in Mesopotamia," *Zeitschrift für Assyriologie und Vorderasiatische Archäologie*, NF Vol. 18 (1957), pp. 91–140), and throughout the history of Ancient Mesopotamia, expressions of popular discontent and even rebellion are to be found; *cf.* B. Brentjes, "Einige Quellen zur Geschichte der Klassenkämpfe in Alten Orient," *Klio*, vol. 46 (Berlin-DDR, 1966), pp. 27–44. One Old Babylonian epic myth contains even a precise description of a violent strike among field workers who have to be bought off, transposed into the world of the Gods (The Story of *Atrahasîs*; see Jacobsen, *The Treasures of Darkness*, pp. 116ff.). But while the philosophical movement in Greece was intimately connected with the political climate of the *agora*, the Mesopotamian scribal culture had no such connections. Heraclitus, himself a descendant of kings and aristocrats, could make eternal movement and change the very basis of his philosphy; in contrast, the scribe telling about a popular rising in Babylon around 1600 B.C. could only consider the ensuing social changes on a par with the catastrophes of a plague which the Gods had to stop (see Brentjes, *op. cit.*, p. 36); moreover, even though the rebellious minor Gods of the *Story of Atrahasîs* are admitted to have been subjected to too harsh suppression while digging the Euphrates and the Tigris, their strike is not given approval by the progress of the story; instead, the instigator of the rebellion is slaughtered and his flesh and blood used for the creation of mankind — and a world order is created where mankind has the duty to take over obediently the toil of the Gods. Thus, political expression on the part of the people was known to the scribes, but it was not tolerated.

On the other hand, the dissimilar social affiliation of literacy seems to make up only part of the difference between Mesopotamia and Greece. As early as the mid-eighth century, the Greek Assembly and *agora* appear to have permitted a sort of free critical speech hardly known in Mesopotamia. Not only are deliberating Assemblies described several times in the Homeric epics (*Iliad* II, 211–293; *Odyssey* I, 26–95 and II, 9–259); after all, such Assemblies where the aristocrats deliberate in front of the people are well known both to historians of Ancient Germania and to anthropologists. But the Thersites episode (*Iliad* II, 211–268) shows that even if the Assembly was definitely no constitutional democratic organ, commoners speaking up against the rulers must have been a well-known phenomenon to the bard's aristocratic public; further, the way Thersites is brought to silence by a blow of irregular kingly violence makes it clear that no regular procedures for doing so were at hand; finally, the presentation of Ulysses' irregular use of his staff as an appeal to the feelings of the public demonstrates that the poet expects it to appreciate the methods by which the ugly commoner was put in his place in the Good Old Days. So, we are told that commoners speaking up against their betters were no only well known but also felt to be a nuisance by the latter; and probably also that those days had already passed away when a commoner could be beaten up in the Assembly for speaking critically.

In *Seven against Thebes*, 5–8, Eteokles complains that in case of disaster the King's name will be "on many a citizen's tongue, bruited up and down in mutterings and laments." It has been argued that this critically minded political attitude was produced by the "hoplite reform" of c. 650 B.C. (first perhaps by Aristotle in *Politica* 1297^{b}22–25; references to modern authors in Lloyd, *Magic, Reason and Experiment*, p. 258, n. 142). The *Iliad* tells us that although the military reform may have enhanced the hoplite infantrymen's criticism of the prevailing aristocratic power, it was not the pressed by Proklos as an increase in the number of theorems,[54] but also a fundamental qualitative change (spoken of by Proklos as a "more scientific arrangement"[55]). Indeed, what happens is a continuation and accentuation of a process inaugurated by Hippocrates of Chios when he wrote the presumably first set of *Elements*. Gradually, mathematics comes to consist of larger, theoretically coherent structures; no longer just reasoned, it becomes deductive and, in the end, axiomatic. The ideals for the organization of mathematical knowledge are clearly delineated by Aristotle in the *Analytica Posteriora*, where these ideals are even put forward as paradigmatic for all scientific knowledge (which, according to Aristotle, consists of deductive derivation from principles).

No doubt the search for coherent structures gave extra impetus to the quantitative growth of knowledge; and probably, the quantitative growth called for better organization. But this internal dynamic of late fifth and early fourth century mathematics was only made possible because mathematics had become something possessing a *social identity*. Mathematical activity had become institutionalized; the very successes of mathematics toward the late fifth century had made it a field of learning of its own.

Mathematics was institutionalized on several levels. It was introduced into the *paideía* of adolescents; but that was in all probability without effect on the dynamics of mathematical knowledge (apart from the recruitment thereby procured), the mathematics taught to adolescents being quite elementary. But mathematics also became something which one might study as a philosopher taught by a teacher — e.g., as one of the "mathematicians around Menaechmos." The increasing systematization of reasoned and argumentative mathematics performed as an autonomous activity automatically led toward deductivity and axiomatization. Systematization relentlessly revealed laws and circularities in argumentation; no mathematician gathering a circle around himself could then avoid trying to get rid of such defects.

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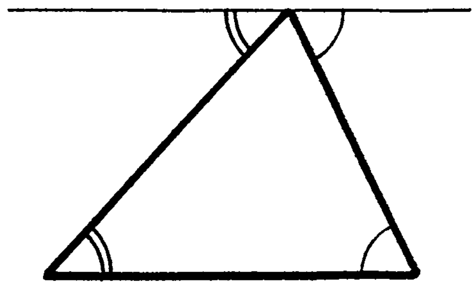
original creator of such criticism — and so, that the political discourse which put its stamp on Greek philosophy has its roots far back in Greek history.

54 Friedlein, *Procli in Primum Euclidis* . . ., p. $66^{16f.}$.

55 "*Eis epistēmonikōtéran sýstasin,*" *ibid.*, $66^{17f.}$.



In the phase of merely reasoned mathematics, it would have been possible to prove that the sum of the angles in a triangle equals two right angles by drawing the parallel without questioning its existence[56]; in other connections, it would be possible to argue for the existence of a parallel in a way which ultimately involved knowledge of the sum of the angles of a triangle. In fact, such attempts to prove everything are discussed by Aristotle in *Analytica posteriora* 72^{b}33–73^{a}20. When geometry became an integrated system, the circularities arising from the combination of such piecemeal demonstrations would become evident. Even this is borne out by Aristotle when he speaks of "those persons . . . who suppose that they are constructing parallel lines: For they fail to see that they are assuming facts which it is impossible to demonstrate unless parallels exist."[57] The recognition would then force itself upon the mathematicians that some things had to be presupposed, in agreement with the initial sentence of the *Analytica posteriora*, that "all instruction given or received by way of argument proceeds from pre-existent knowledge" (71^{a}1–2). In the end, Euclid found a way out of the complex problem by a combination of his fifth postulate (which implies that at most one parallel exists) and a tacit assumption about the figure used to prove Prop. I.16, an assumption which holds true only in geometries where parallels exist (i.e., not on a sphere).[58]

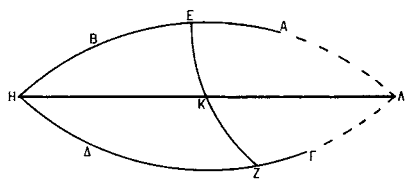
56 According to Proclos (*ibid.*, p. 379$^{2ff.}$), such a proof is ascribed by Eudemos to "the Pythagoreans." Knorr, *The Evolution* . . ., shows how this ascription *may* have arisen by a misunderstanding. Of course, it may also be genuine.

57 *Analytica priora*, 65^{a}4–7.

58 The fifth postulate of the *Elements* tells us "that, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles." Proposition I.16 claims that "in any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles" (both quoted from *The Thirteen Books of Euclid's Elements*, translated with introduction and commentary by Sir Thomas L. Heath. I–III. Revised ed.,

Trying to describe the character of the mathematical discourse of this second phase, we can say that it becomes closed into itself, i.e., autonomous: mathematics builds up its own system of scientific and epistemological norms; when Protagoras argues against the geometricians that "a hoop touches a straight edge not at a point [but along a stretch of finite length],"[59] he just disqualifies himself (in the eyes of fourth-century geometricians) by being unaware of this closing of mathematical discourse to nonmathematical reason. In the elementary *paideía*, the mathematical discourse also becomes closed in the sense of being one-way, dependent on authority and open to no alternative thinking. The same process is on its way at the philosophical level, but here only as a goal pursued: Mathematics is understood as concerned with eternal, immovable truth, and thus it cannot admit of alternatives and discussion of its foundations; so, mathematics must by necessity *aim* at the closure of its own discourse.[60]

Cambridge and New York, 1926, reprint New York, 1956). This is used to prove proposition I.27, "if a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another," by means of which a parallel to a given line can be constructed.



I.27 follows, because the two lines AB and ΓΔ if not parallel will produce a triangle HEZ together with the transversal, in which case the angle ΓZE will be greater than the angle BEZ, according to I.16. However, to prove I.16, the mid-point K of EZ is constructed, and HK is produced to Λ, where KΛ = EK. The proof assumes that Λ must be situated between the lines HA and HΓ. But on a sphere Λ will be the other intersection of the two lines, and the proof of I.16 becomes invalid.

59 Cited from Aristotle, *Metaphysica* 998ᵃ1–4, who quotes Protagoras' polemics with the geometers to make clear that even the geometry of the astronomers deals with something different from perceptible points, lines and circles. The autonomy of mathematics appears to be the paradigmatic case for Aristotle's general idea of autonomous single sciences.

60 Curiously enough, this attempt to close discourse in the "Platonic reform" of mathematics (a term coined by Zeuthen) did not prevent Plato from advocating "discovery learning" in mathematics for children, as can be seen in his description of Egyptian arithmetic teaching in *The Laws* VII, 819. If truth itself is the authority, the teacher can do without authoritarian methods. Only the imposition of socially necessary *lies* requires the cruel measures proposed toward the end of the same work (X, 907ff); *cf.*

By the end of our second period, this process was carried to its end. Already by the end of the fourth century, "a common fund of theorems existed, in plane as well as spherical geometry, which had already taken on their definitive form, and the formulation of which was to be perpetuated for centuries with no change whatever."[61] In the early third century B.C. (or, if recent proposals to make Euclid a contemporary of Archimedes are correct,[62] around 250 B.C.), Greek mathematical thought had been shaped in the Euclidean *Elements* as a specialized, hard, sharp and immensely effective tool for the production of new knowledge. In the third period, it produced astonishing quantitative accretions, e.g., in the works of Archimedes and Apollonios. But its whole style and formal character was fixed. It was deductive, axiomatic, abstract and formally "pure"; and it was totally "euclidized." Commentators like Pappos, Proklos and Theon of Alexandria explain and extend (for good or for bad); *grosso modo* they raise no doubts. Truly, Archimedes extended the range of aims of mathematics by his numerical measurement of the circle, in a way which (in spite of its "pure" form) was noticed as a deviation by some commentators.[63] But even if his results were adopted, they inspired no further renewal (apart from the Heronic tradition, the connections of which to Archimedes are clear, but which misses on the other hand the high level of mainstream Greek mathematics). Greek mathematics had, in the *Elements*, got a paradigm in Kuhn's original sense of that word: a book "that all the practitioners of a given field knew intimately and admired, achievements on which they modeled their own research and which provided them with a measure of their own accomplishment."[64] Mathematical discourse now became really closed, in agreement with the "Platonic" intentions; the closing was, however, not effected by a teaching institution, but through

B. Farrington, "Prometheus Bound: Government and Science in Classical Antiquity," *Science & Society*, Vol. II (1937/38), pp. 435–447, esp. pp. 437f.

61 Aujac, "Autolycos de Pitané, prédecesseur d'Euclide," p. 10 (see above n. 28).

62 See I. Schneider, *Archimedes: Ingenieur, Naturwissenschaftler und Mathematiker* (Darmstadt, 1979), pp. 61f.

63 See p. 362 in K. Vogel, "Beiträge zur griechischen Logistik. Erster Theil," *Sitzungsberichte der mathematisch-naturwissenschaftlichen Abteilung der Bayerischen Akademie der Wissenschaften zu München*, 1936, pp. 357–472.

64 P. 352 in Thomas Kuhn, "The Function of Dogma in Scientific Research," pp. 347–369 in A.C. Crombie (ed.), *Scientific Change* (London, 1963).

a book — or, better, it was effected by a teaching institution whose main institutional aspect was the use of that book.

This institutionalization did not give Greek mathematics of the mature phase any *new* character. But it was the precondition for the perpetuation of a character which Greek mathematics had once acquired through a series of settings which had now disappeared: institutionalized and non-institutionalized, discursively open and discursively closed.

The social carriers of mathematical development in the mature phase had less to do with teaching, institutionalized or not, than their predecessors of any earlier phase. Only a few, and not the greatest, were connected to those institutions of higher learning which, from Plato's Academy onwards, had succeeded the earlier philosophical circles or informal "schools." Most great mathematicians are best described as "professional full- or part-time amateurs" (strange as this mixing may sound to modern sociological ears), who after a juvenile stay, e.g., at the centers of higher learning in Alexandria remained in mutual contact through letters, and whose impregnation with the professional ideals of the discipline early in life had been so strong that their lonely work and their letters could maintain them as members of one stable scientific community.

III. *The Latin Middle Ages: A Discourse of Relics*

In order to prevent any simplistic — or just simple — picture from emerging, I shall discuss one more episode. I shall bypass the very interesting relations between socioeconomic background, traditions and institutions of teaching, types of mathematical discourse, and the development of mathematics in Ancient and Medieval India and in the Medieval Islamic world.[65] Instead, I shall concentrate on the "Latin Middle Ages" of Western Europe.[66]

The Roman part of the Ancient world had never shared the Greek interest in theoretical mathematics; as Cicero remarks, the Romans restricted their mathematical interests to surveying and

65 Some hints concerning India and a somewhat more elaborated sketch concerning Islam will be found in my "Influences . . . ," pp. 52–61, 108–113.

66 Once more, factual background information is found in my "Influences . . .," pp. 62–77, 113–115. Supplementary background is supplied by my "Jordanus de Nemore . . . " (above, nn. 5 and 22).

calculation.[67] Truly, the education of a Roman gentleman was built on the seven "Liberal Arts" of the Greek *paideía*: grammar, rhetorics, dialectics (i.e., logic), arithmetic, geometry, harmonics ("music"), and astronomy. But what was taught in the latter four mathematical arts was utterly restricted. Only a few of the Greek mathematical works were ever translated into Latin (only one of which, part of the *Elements* translated by Boethius around A.D. 500, was of scientific merit), and only superficial popularizations were ever written directly in Latin.

The Christian takeover of education in Late Antiquity did nothing to repair this: on the contrary. Still more unquestionably than the gentleman, a good Christian should definitely not be secularly learned, even though good manners required him to be culturally polished. The breakdown of the Western Empire and the rise of loose barbarian states deprived him even of the possibility to be polished.

Still, if not saving much of the Ancient learned legacy, the Christian Church saved at least the faint memory that *something had been lost*. At every occasion of a cultural revival, be it visigothic Spain in the early 7th century, Anglo-Saxon England in the early 8th, the Frankish Kingdom of the early 9th or the Ottonian Empire of the late 10th, the recurrent characteristic is an attempt to reconquer the cultural ground which had been lost. This is the reason why every revival looks like a "renaissance" and has been labeled so by its modern historians.

Until the end of the first millenium, the reconquests of the mathematical front were restricted to arithmetical Easter calculation ("computus"); Ancient presentations of theoretical arithmetic and harmonics for non-mathematicians; and some surveying manuals which had to play the role of geometry. Now, by the onset of the High Middle Ages, things were going to change. But in the actual moment, mathematics was (like every remnant of Ancient culture) as much a sacred relic as something to be learned and understood or a type of discourse; furthermore, even as a subject to be learned or as discourse, mathematics was

67 "By them [i.e., the Greeks], geometry was in the highest honour; therefore nobody was more illustrious than the mathematicians: but *we* have restricted this art to the practical purposes of measuring and reckoning." *Tusculan Disputations* I,ii,§ 5 (Loeb Classical Library; Cambridge, MA, and London, 1971; 1st ed., 1927).

profoundly marked by being a relic in a culture given to worshipping relics.[68]

An economic and demographic leap forward in the 11th and 12th centuries was the occasion for a revival of trade and monetary economy and for the rise of towns achieving a certain degree of autonomy (*de facto* juridical autonomy, and often even officially recognized juridical autonomy). Men participated in the social and political life of these towns as members of more or less institutionalized groups, inside which they acted as equals; presumably it was on this background that an interest in open reasoned discourse grew in the 11th century urban environment.[69]

The economic revival was also the occasion for the growth of *cathedral schools*, the students of which were taught the seven Liberal Arts to the extent that competent teachers and the necessary text materials were at hand. From the point of view of the Church, the schools were designed to provide future priests, etc., with the knowledge necessary in a new social context where the priest had to be more than the main actor of rituals and the magician of relics.[70] At the same time, the growth of the schools can be seen as yet another spontaneous expression of the recurrent tendency to translate cultural blossomings into "renaissances," revivals of Ancient learning. Finally, the clerks trained at the cathedral schools would often come to serve in and outside the Church in cancellarian and secretarial functions, the

68 A striking example is Gerbert's joyful announcement (c. 983) to his friend Adalbero that he has found "8 volumes: Boethius' *On Astronomy*, some utterly splendid [volumes] with geometrical figures, and some others not less to be admired"; see N. Bubnov (ed.), *Gerberti postea Silvestri II papae Opera Mathematica (972–1003)* (Berlin, 1899), pp. 99–101 (including n. 6).

69 In 1090, we are told by Hermann of Tournai, the squares of his city were filled by curious crowds when Master Odo discussed philosophical questions with his students, and "that the citizens left their various employments so that you might think them fully attached to philosophy." See pp. 57 and 93, n. 358 in E. Werner, "Stadtluft macht frei: Frühscholastik und bürgerliche Emancipation in der ersten Hälfte des 12. Jahrhundersts," *Sitzungsberichte der sächsischen Akademie der Wissenschaften zu Leipzig, Philologisch-historische Klasse*, Vol. 118, Heft 5 (Berlin-DDR, 1976). This eminent monograph is fundamental to the understanding of the rationalism of the 11th and 12th centuries, its background and its implications. Briefer statements in the same avowedly historical materialist spirit were formulated by the Dominican Father M.-D. Chenu: "Praxis historique et relation eglise-sociétè," *La Pensée*, N° 192 (Avril 1977), pp. 5–11; and "Civilisation urbaine et théologie," *Annales E.S.C.*, Vol. 29 (1974), pp. 1253–1263.

70 See P. Mandonnet, O.P., "La crise scolaire au début du XIII^e^ siècle et la fondation de l'ordre des Frères-Prêcheurs," *Revue d'histoire ecclésiastique*, Vol. 15 (1914), 34–49, on the persistent problem presented by uneducated priests.

non-engineering aspect of the old scribal function. So long, however, as the ideological interest in free discourse and the cultural need for a "renaissance" prevailed, the schools did not take on the character of scribal training schools. The disciplines of autonomous thought, which had not been known to the Babylonians, were now at hand, and they were fundamental to the scholars' conception of their world and of their own identity.

For mathematics, the 11th century school meant little directly, apart from a firmer possession of the insecure reconquests of the late first millenium. An awakening interest in astrology, nurtured by a few translations of Arabic treatises on the subject in the 10th and 11th centuries,[71] is probably best understood as an expression of the search for natural explanations distancing direct Divine intervention,[72] an essential search in a society on the way to rationalize its world picture. Even though partly carried by cathedral school masters, it was no product of the school institution as such, whose whole heritage from both Ancient learning and from the Fathers of the Church was to separate liberal-arts astronomy from astrology.[73]

Indirectly, though, the 11th century school had great importance for the future of mathematics. Taken together, the rational discursive spirit of the times and the training and opportunities provided by the schools gave rise to great changes in every corner of Latin learning. On mainly native ground, figures

71 J.W. Thompson, "The Introduction of Arabic Science into Lorraine in the Tenth Century," *Isis*, Vol. 12 (1929), pp. 184–193; and A. van der Vyver, "Les plus anciennes traductions latines médiévales (X^{e}–XI^{e} siècles) de traités d'astronomie et d'astrologie," *Osiris*, Vol. 1 (1936), 658–691.

72 Related interpretations are discussed L. Thorndike, "True Place of Astrology in the History of Science," *Isis*, vol. 46 (1955), pp. 273–78; R. Lemay, *Abu Ma'shar and Latin Aristotelianism in the Twelfth Century: The Recovery of Aristotle's Natural Philosophy through Arabic Astrology* (Beirut, 1962), pp. xxiiff and *passim*; A. Birkenmajer, "Le rôle joué par les médecins et les naturalistes dans la réception d'Aristote au XII-e et XIII-e siècles," reprint in Birkenmajer, *Études d'histoire des sciences et de la philosophie du Moyen Âge* (Studia Copernicana I; Wrocaw, Warszawa and Kraków, 1970), pp. 73–87; and Tullio Gregory, "La nouvelle idée de nature et de savoir scientifique au XIIe siècle," pp. 193–218 in J.E. Murdoch and E. Sylla (eds.), *The Cultural Context of Medieval Learning* (Boston Studies in the Philosophy of Science; Dordrecht and Boston, 1975), especially pp. 203ff.

73 See Augustine, *De civitate Dei* V, i–viii (*Concerning the City of God, against the Pagans*, Harmondsworth, Middlesex, 1972); Isidor of Sevilla, *Etymologiae* III, xxvii (J.-P. Migne (ed.), *Patrologiae cursus completus, Series Latina*, Paris, 1844–1864; vol. 82); and Hrabanus Maurus, *De clericorum institutione* III, xxv (Migne, *Patrologia latina*, Vol. 107).

like Anselm of Canterbury, Abaelard, Hugh of Saint-Victor, the "12th century Platonists" and Gratian recast philosophy, theology and Canon Law in the late 11th through the mid-12th century. More important for mathematics, the background provided by the schools made possible the translation of Muslim learning and of Arabic versions of Ancient Greek works (and, initially to a lesser extent, direct translation of Greek works) by creating the scholarly competence and not least the enormous enthusiasm of the translators; furthermore, the schools provided a public which could receive the translations and have them spread. So, during the 12th century, most main works of Ancient and Judeo-Muslim astrology (including Ptolemy's *Almagest*), the *Elements*, al-Khwârizmî's *Algebra*, and several expositions of "Hindu reckoning" (the decimal place value system for integers and its algorithms) were, together with many other mathematical and non-mathematical works, translated and spread.[74] Even the larger parts of Aristotle's *Organon*, the *Metaphysica*, and part of his natural philosophy were first transplanted in the mid to late 12th century.

In the late 12th century, learning at those cathedral schools which were to develop into universities was in a situation of suspense. Learning was not seen as something being created in an active process. Learning already existed in the form of great scholarly works, "authorities"[75]; the enthusiasm for knowledge still had something of the enthusiasm for relics.[76] And yet, the "new learning" was really something *new*; as a body of relatively

74 The basic information on the translations will be found in M. Steinschneider, "Die europäischen Übersetzungen aus dem arabischen bis Mitte des 17. Jahrhunderts," *Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften in Wien, philosophisch-historische Klasse*, Vols. CXLIX/iv (1904) and CLI/i (1905) (reprint Graz, 1956); and in Ch. H. Haskins, *Studies in the History of Mediaeval Science* (Cambridge, MA, 1924). *Cf.* also M. Steinschneider, *Die hebraeischen Übersetzungen des Mittelalters und die Juden als Dolmetscher* (Berlin, 1893; reprint Graz, 1956).

75 It is characteristic that the Middle Ages conflated the two Latin words *auctor*, "source of a written text," and *auctoritas*, "source of power."

76 This becomes particularly clear in the case of the translations of Greek authors made directly from the Greek, which, in Haskin's words (*op. cit.*, pp. 150–152) were "closely, even painfully literal, in a way to suggest the stumbling and conscientious schoolboy. Every Greek word had to be represented by a Latin equivalent. Sarrazin laments that he cannot render phrases introduced by the article, and even attempts to imitate Greek compounds by running Latin words together. . . . This method, *de verbo ad verbum*, was, however, followed not out of ignorance but out of set purpose, as Burgundio, for example, is at pains to explain in one of his prefaces. The texts which

coherent knowledge it was in fact something which was being actively created. In spite of its concentration on already existing authority, the discourse of the new learning was anything but closed: There were still tight connections between the open "political" discourse of the corporations ("universities") of masters and students and the discourse of learning.[77] The "political" discourse of the scholars, on its part, was of the same genre as that which had manifested itself in the turbulent urban environment already in the later 11th century, reinforced in a synergetic process by the increasingly "dialectical" organization of the teaching institution.

The openness and the semi-political character of the learned discourse of the late 12th and early 13th century was not mistaken at the time. As Bernard of Clairvaux had attacked the rationalizing theology of Abelard and the "Platonist" philosophers, so many smaller theological minds attacked the new learning, complaining, e.g., that the Christians (and even monks and canons) not only wasted their time but also endangered their salvation studying

> the [grammatical] rules of Priscian, the Laws of Justinian, the doctrine of Galen, the speeches of the rhetors, the ambiguities of Aristotle, the theorems of Euclid, and the conjectures of Ptolemy. Indeed, the so-called liberal arts are valuable for sharpening the genius and the understanding of the Scriptures; but together with the philosopher [i.e., Aristotle] they are to be saluted only from the doorstep. . . . Therefore, the reading of the letters of the Pagans does not illuminate the mind, but obscures it.[78]

It will be observed that even interest in the *Elements* is presented

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these scholars rendered were authorities in a sense that the modern word has lost, and their words were not to be trifled with. Who was Aristippus that he should omit any of the sacred words of Plato? Better carry over a word like *didascalia* than run any chance of altering the meaning of Aristotle. Burgundio might even be in danger of heresy if he put anything of his own instead of the very words of Chrysostom." In the preface mentioned, Burgundio speaks literally of his translation of Chrysostomos as *scriptura sancta* (*ibid.*, p. 151, n. 36).

77 In a way suggesting a (freely invented) student couple of 1968 giving the name *Che* to a son, Abaelard and Héloïse called their little son *Astrolabius*, by the name of the main astronomical observational and calculational device (Abaelard, *Historia calamitatum*; p. 184f. ed. J.T. Muckle, C.S.B., *Mediaeval Studies*, Vol. 12 (1950), 163–213).

78 Stephen of Tournai, quoted from M. Grabmann, "I divieti ecclesiastici di Aristotele sotte Innocenzo III e Gregorio IX," *Miscellanea Historiae Pontificiae edita a Facultate Historiae Ecclesiasticae in Pontificiae Universitate Gregoriana*, Vol. V/1 (Rome, 1941), p.

as a danger to theological order (and, behind that, to social order in the ecclesiastical universe).

That, however, was only for a brief period. Euclidean mathematics was not fit to serve the construction of a coherent counterdiscourse to the discourse of the conservative theologians. Interest in mathematics dropped back from the front in 13th century learning. The great conflicts, the prohibition of dangerous works, and the executions of heretical scholars were all concerned with Aristotelian philosophy and its derivations (including pseudo-Aristotelian occult science).[79] Except for a few active researchers, the scholars of the 13th century looked at mathematics as a venerated part of the cultural heritage. In the context of the 13th century university world, mathematics was better fitted as a modest part of the peaceful synthesis than as a provider of revolutionary counterdiscourse.

Such synthesis did take place in the mid-13th century. It took place at the social and political level, where the liberation movement of the towns attained a state of equilibrium with the prevailing princely power, and where even princely, papal and feudal power learned to live together; and it was seen at the level of religious organization, where the specific urban spirituality got an authorized expression through the orders of friars, but where it lost its *autonomous* expression through lay pauper movements. Closer to our subject, synthesis forced its way through in the matter of Aristotelian philosophy. By the 1230s, the position of the conservative theologians had become untenable: Aristotelian metaphysics had penetrated their own argumentation to the bone. But for all the fluidity and turbulence of the university environment, general social and ideological conditions were not ripe to overthrow the power of the ecclesiastical institution. This scholarly stalemate was solved through the great Christian-Aristotelian synthesis due to Saint Thomas and Albert the Great. Thanks to their work, the "new learning" was fitted into the world conception of Latin Christianity, as that cornerstone which had been missing (and missed) for so long. One is tempted to

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61. On p. 60, Grabmann quotes Absalon of Springersbach (like Stephen late 12th century) to the effect that "the spirit of Christ reigns not where Aristotle's spirit is Lord."

79 See Grabmann, *op. cit.*; and F. van Steenberghen, *La philosophie au XIII^e siècle* (Louvain and Paris, 1966).

turn St. Thomas' famous dictum[80] around, to the effect that "natural philosophy did not abolish the world view of Divine Grace; it made it complete."

By making it complete, however, Latin Aristotelianism was given an orthodox interpretation which deprived it of its character of open discourse. Only for a short time (the 14th century) was it able to develop answers to new questions, and to procure a world view which could really pretend to being a view of a world in change; after that, Aristotelian learning (and the whole of Medieval University learning, which had come under its sway) was upheld only by institutional inertia and by the external social forces guaranteeing its survival. Gradually, the universities were to develop into training schools for priests, lawyers, physicians and officials, of scribal-school character. Their much-beloved dialectical method, once a reflection of the open critical discourse, could be derided as a display of empty virtuosity by the satirical authors of later ages, from Thomas More to the 18th century.

Mathematics was no main pillar in the synthesis. But on a modest level, it had prepared the way. Neither the mathematics of the "new learning," nor mathematics at all or its single constituent disciplines, were ever practiced on a larger scale as something autonomous. The traditional mathematical disciplines belonged to the total scheme of Liberal Arts. The commentaries written into mathematical treatises, be it the 11th century low-level presentations of theoretical arithmetic[81] or the 12th- and 13th-century *Elements*[82] demonstrate their attachment to a teaching tradition where the establishment of connections to the non-mathematical disciplines of the curriculum were as important as mathematics itself. These traditional disciplines remained — the short-lived tendencies of the 12th century and a few mathematicians by inclination in the following centuries[83] disregarded — integrated parts of a larger cultural whole, and parts of a herit-

80 *Gratia non Tollit naturam sed perficit.*

81 See G.R. Evans, "Introductions to Boethius's 'Arithmetica' of the Tenth to the Fourteenth Century," *History of Science*, Vol. 16 (1978), pp. 22–41.

82 See, e.g., J.E. Murdoch, "The Medieval Euclid: Salient Aspects of the Translations of the *Elements* by Adelard of Bath and Campanus of Novara," *Revue de Synthèse*, Vol. 89 (1968), pp. 67–94.

83 The works of Jordanus of Nemore, the greatest (or only really great) traditional mathematician of the Latin 13th century, are analyzed together with the reasons for his relative isolation in my "Jordanus de Nemore . . ." (Leonardo Fibonacci was not a "traditional" mathematician, in a sense referring to the Medieval Western tradition.)

age. Mathematical discourse was never, as it had been in Antiquity from the fourth century B.C. onwards, autonomously closed on itself; its main epistemological responsibility was not inwardly but outwardly directed. But it was closed on past performances, closed to fundamental renewal and closed to alternatives.

The non-traditional disciplines were necessarily less closed: Algebra, optics, "science of weights" (i.e., mathematical statics) did not fit into the traditional curriculum and could only be merged with it through a creative process. But only a handful of people engaged in these disciplines, and of the works of that handful of scholarly eccentrics only those that were congruous with traditional thought gained general acceptance.[84]

The prevailing scholarly synthesis was a real synthesis of all the important interests present: Only few of those concerned did not feel well inside its frame. Those who did not (and their number was increasing as the 14th century approached and especially during its progress) would rather leave the universe of closed rational discourse altogether and fall into mysticism and skepticism, than try to open it to alternative rationalities — be they mathematical or philosophical.

The late 13th and the 14th centuries offer what might appear to be a pair of exceptions to this generalization: A violent increase in the semi-rational, semi-mystical field of astrology; and an attempt at mathematization of Aristotelian natural philosophy (and other fields which had taken over the Aristotelian conceptual structure), through a quantification of "movement" and "qualities."[85] Neither, however, led to the establishment of lasting mathematical disciplines.[86] Instead, the mathematical renewal of the Renaissance came from elsewhere: first, from the "abacus schools" of Northern Italy, training schools for merchant youth, spiritually (and perhaps also genetically, by way of

84 Of Witelo's *Perspectiva*, Book I was introduced into many university curricula as an alternative to the *Elements*. This is the part of the work which deals *not* with optics but with general geometry; see S. Unguru (ed., transl.), *Witelonis Perspectivae liber primus* (Studia Copernicana XV, Wrocław, Warszawa and Kraków, 1977). *Cf.* also above, n. 22.

85 See, e.g., the discussion of the current in J.E. Murdoch, "*Mathesis in philosophiam scholasticam introducta*: The Rise and Development of Mathematics in Fourteenth Century Philosophy and Theology," in *Arts libéraux et philosophie au Moyen Âge*. Actes du quatrième congrès international de philosophie médiévale, 27 août–2 septembre 1967 (Montréal and Paris, 1969), pp. 215–254.

86 And, in contrast to what happened in the Islamic world (where astronomical teaching

schools and teaching traditions of the Islamic world) related to the Old Babylonian training schools for scribes[87]; and second, from alternative approaches to the mathematical disciplines of Antiquity inspired by the humanist movement and by the problems of perspective painting and architecture.[88] The real explosion took place in the confrontation of these two sources for renewal, e.g., in the famous episode[89] when Cardano published Tartaglia's solution to the third-degree equation (giving "due credit"), thereby bringing the secret method which had made Tartaglia a virtuoso (in the *genre* of a traditional abacus school master or an Old Babylonian scribal teacher[90]) into the light of open scientific discourse.

IV. *Perspectives*

The third episode ends when the merging of traditions and the reorganization of all learned and literate discourse made possible by printing opened the ways to modern mathematics. Much has changed since then. Much has changed, even concern-

was organized in other patterns; *cf.* my "Influences . . .," pp. 60f.), the interest in astrology gave rise to little mathematical activity beyond the writing of compendia.

87 See, e.g., A. Fanfani, "Le préparation intellectuelle à l'activité économique, en Italie, du XIV[e] au XVI[e] siècle," *Le Moyen Âge*, Vol. 57 (1951), 327–346; a→ Ch.T. Davis, "Education in Dante's Florence," *Speculum*, Vol. 40 (1965), 415–435. Leonardo Fibonacci is one link to Islam, but not the only one.

88 *Cf.* P.L. Rose, *The Italian Renaissance of Mathematics. Studies on Humanists and Mathematicians from Petrarch to Galileo* (Genève, 1975), pp. 26–56 and *passim*, on the relations between the humanist movement and mathematics; and M.D. Davis, *Piero della Francesca's Mathematical Treatises* (Ravenna, 1977), pp. 1–20, on the teaching of artists' and architects' mathematics in the abacus school.

89 Described in some detail by Ore in Girolamo Cardano, *The Great Art or The Rules of Alegbra*, translated and edited by T.R. Witmer (Cambridge, MA, 1968), pp. xviii–xxii.

90 The parallel can be carried into mathematical details. As mentioned above, higher-degree equations occur in the Old Babylonian texts in a way which suggests their exclusive use for the display of fake professorial virtuosity. Similarly, the abacus masters showed their ability by solving such equations. But "the rules given [by Piero della Francesca, abacus master and painter] for solving equations of the third, fourth and fifth degrees are valid only for special cases of these equations. The rule for solving the equation of the sixth degree is altogether false" (S.A. Jayawardene, "The 'Trattato d'abaco' of Piero della Francesca," p. 243; in C.H. Clough, ed., *Cultural Aspects of the Italian Renaissance. Essays in Honour of Paul Oskar Kristeller* (Manchester and New York, 1976), pp. 229–243). The virtuosity displayed by Piero was at least as fake as that of the scribe school teachers, and Tartaglia and del Ferro (who found the solution first) could claim to be the first real virtuosos in a field where imposture had reigned for 3500 years.

ing the contents, structure, obligations and social organization of mathematics. If *any* key science can be pointed out in the ongoing scientific-technological revolution, mathematics is the key. Indeed, a recurrent condition for the integration of the theoretical sciences with the management of practical affairs is an analytical reformulation of theories which either amounts to mathematization or opens the way to mathematization — eventually through the development of new mathematical disciplines which serve that mathematization.[91] So, mathematics and mathematicians must be held co-responsible, both for the fulfillment of the promises of that revolution and for its dangers and perversions.

If this is so, what then is the use of a discussion of the historical development of mathematics as a system of organized knowledge and as a discourse stopping around A.D. 1500 — be it inspired from historico-materialist points of view? Does it have indications beyond itself (500 years beyond itself!), or is it merely a piece of historical research practiced as pure, immaculate science?

I think the story has important implications for the present, politically as well as pedagogically. Politically, it suggests that the concepts of "open" and "closed" discourse are central,[92] but it makes no historical (and hence, when applied to the present, no political) sense to posit a mechanistic and abstract dichotomy, in the manner of Karl Popper, between the Open Society and the Open Discourse, on the one hand, and the Closed Society and the Closed Discourse, on the other. Things are more complicated than that, and therefore the relations between openness and closedness must be understood in their social concreteness, not accepted as abstract postulates. This is as true today as in the pre-Renaissance era.

91 A somewhat more detailed discussion of the relations between contemporary mathematical development and the scientific-technological revolution will be found in B. Booss & J. Høyrup, *Von Mathematik und Krieg* (Schriftenreihe Wissenschaft und Frieden, No. 1; Marburg, 1984), pp. 18–44, *passim*.

92 In this respect, the story can be regarded as an elaboration of an important theme from the anti-fascist historical sociology of science, as it was dealt with, e.g., in Farrington, "Prometheus Bound . . ." (above, n. 60); H.E. Sigerist, "Science and Democracy," *Science & Society*, Vol. II (1937–38), pp. 291–299; and R.K. Merton, "Science and the Social Order" (1937) and "Science and Technology in a Democratic Order," both reprinted, e.g., in Merton, *Social Theory and Social Structure* (enlarged edition; New York and London, 1968), pp. 591–615.

The pedagogical implications of the story concern the obligations of mathematics education. The contemporary world is confronted with immense problems: problems of survival, and problems of welfare. Disregarding the dangers of nuclear war, how shall we avoid climatic and ecological catastrophes and the wasting of resources necessary for the future of civilization; how shall we secure for all human beings a life in dignity; how shall we provide everybody with food, housing and leisure? These are not only mathematical and technological problems, and our present capitalist order may well be as unable as it is unwilling to solve them at all. But they are *also* technological and mathematical — *no* social order will solve them without using adequate technologies — *and many of the necessary technologies have not yet been developed, nor has the mathematics required for their development been shaped.* Therefore, mathematics must be allowed to develop, and mathematical development furthered to the greatest extent possible; and historical experience suggests that mathematics must be allowed the autonomous existence required for that.

On the other hand, many of the problems of our contemporary world are due to unconscious, restricted technological and scientific (sub-)rationality (behind which stands of course political interests and economic structures favoring the development and perpetuation of unconscious and unscrupulous pseudo-rationality). The acceptance of this situation, the belief that such rationality is the only possible and perhaps genuine rationality is supported by the appearances of technology, science *and mathematics* as they present themselves in their current closed discourses. If the transmission of mathematics to the next generation is to contribute to making a better world, these closed and self-sufficient discourses must be combatted from the inside; mathematics must be made known also under the aspect of an open discourse, and as an integrated part of human knowledge.

It must be the duty not least of the teacher, from primary school to university, to present a double picture of mathematics to students: as a field of human knowledge with its own *integrity*, requiring its own autonomous further development; but which at the same time must achieve *integration* as part of human knowledge and human life in general.

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